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The heave added-mass and damping coefficients of a submerged torus

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This paper describes the calculation of the added-mass and damping coefficients of a submerged toroidal body that is undergoing a forced, periodic heaving motion. The velocity potential of the motion is expressed as an infinite sum of toroidal multipole potentials, and the problem is solved in a manner analogous to Ursell's classical solution for a submerged circular cylinder in two dimensions. When the torus is 'slender', in the sense that its tubular radius is small compared with its overall diameter, relatively simple closed-form asymptotic approximations for the added-mass and damping coefficients are obtained. This work is motivated by the proposed RS-35 design of ring-hulled semisubmersible platform.

1. Introduction

The increasing needs of the offshore-oil industry for exploration, drilling and early production systems to work in the hostile environment north of the 62nd parallel in the North Sea has led to the development of a new type of semisubmersible platform, known as the RS-35 (see *The Naval Architect* 1980). The unique design concept of the RS-35 is of a uniform and well-balanced submerged toroidal ring-hull, which supports a working platform above the water line by means of four vertical columns. It is believed that this toroidal ring-hull design has excellent wave-response characteristics and will allow the search for oil and gas to proceed into deeper and rougher waters than has been previously possible. To give some idea of the scale of this structure, the ring-hull has an overall diameter of about 100 m, each tubular section has a diameter of about 10 m and the vertical supporting columns have a diameter of about 12 m. In its operational mode the ring-hull is submerged to a depth of about 20 m.

It is clearly important to be able to predict the hydrodynamic characteristics of this type of vessel in the presence of waves. In many problems involving a large body in an ocean, a realistic mathematical model can be achieved by assuming that the fluid is inviscid and incompressible and that its motion is irrotational. This leads to the classical description of the fluid motion in terms of a velocity potential that satisfies Laplace's equation in the bulk of the fluid. When the wave amplitude is small compared with its wavelength, and to the dimensions of the body, it is also appropriate to adopt the linearized form of the free-surface boundary condition. In such a linearized theory, the response of a body to incident waves can be determined if we know the added-mass and damping coefficients associated with the forced motions of that body in the absence of waves (see e.g. Newman 1977*a*).

From an engineering viewpoint, it is important to estimate the heave (vertical) motions of the semisubmersible platform due to the influence of ocean waves. This is because the drilling pipes employed in deep waters can be hundreds of metres in length, and so can withstand the small degree of flexure introduced by any surging (horizontal) or rolling motions of the platform. However, the drilling pipe is largely inextensible, and can be damaged by any excessive heaving motion of the platform.

In this paper we examine the problem of calculating the added-mass and damping coefficients of a submerged torus that is undergoing a forced, periodic heaving motion: the so-called 'heave radiation problem'. By introducing suitably chosen toroidal coordinates we will show that the associated boundary-value problems can be formulated and solved in a manner that is analogous to Ursell's classical solution for the waves generated by a submerged circular cylinder in two dimensions (see Ursell 1950).

When the torus is 'slender', in the sense that the tubular radius is small compared with its overall diameter, we will obtain simple closed-form asymptotic approximations for the added-mass and damping coefficients. These approximations are relevant to the proposed RS-35 ring-hull design of semisubmersible.

The author is not aware of any previous work in which the submerged-torus problem is treated by multipole methods. The case of a *half-immersed* slender torus has been discussed by Newman (1977*b*), who used a 'strip-theory' approach to derive approximations for the added-mass and damping coefficients. This strip-theory calculation is only appropriate at high frequencies. An alternative high-frequency approximation has been given by Davis (1975) for the case of a general (i.e. non-slender) half-immersed torus.

Finally, it should be noted that the general methods presented in this paper could also be used to treat the non-axisymmetric modes of motion which have a $\cos m\theta$ variation, where θ is the azimuthal angle.

2. Statement of the problem: definition of toroidal coordinates

To fix ideas, let us consider the origin of cylindrical polar coordinates (r, θ, y) to be in the mean free surface of the fluid, with the y-axis vertical (y increasing with depth) and r, θ taken in the usual way. In these coordinates, the surface $\mathscr{S}$ of the torus is given by

$$\mathscr{S}: (r-c)^2+(y-d)^2 = b^2, \quad 0 \leqslant \theta \leqslant 2\pi \quad (0 < b < c;\ d > b), \tag{2.1}$$

where the geometrical significance of b, c and d is shown in figure 1. For obvious reasons, we will refer to d as the 'depth of submergence' of the torus, b as the 'tubular-radius' of the torus and $2c$ as being the 'overall diameter'.

The surrounding fluid is assumed to be inviscid and incompressible and the motion is assumed to be irrotational. This leads to a description of its velocity field $\boldsymbol{u}(\boldsymbol{r}, t)$ in terms of a velocity potential $\Phi(\boldsymbol{r}, t)$, where

$$\boldsymbol{u} = \nabla\Phi.$$

Waves are generated in the fluid due to a forced, heaving, oscillatory motion of the torus, whose instantaneous (downward) velocity is $U\cos\omega t$. The fluid is assumed to have attained a 'steady state' in which its variation with time is also harmonic, and we can write

$$\Phi(\boldsymbol{r}, t) = \mathrm{Re}\{\phi(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t}\},$$

where $\phi(\boldsymbol{r})$ is a complex-valued potential, to be determined. The equation of continuity in the bulk of the fluid is

$$\nabla^2\phi = 0, \tag{2.2}$$

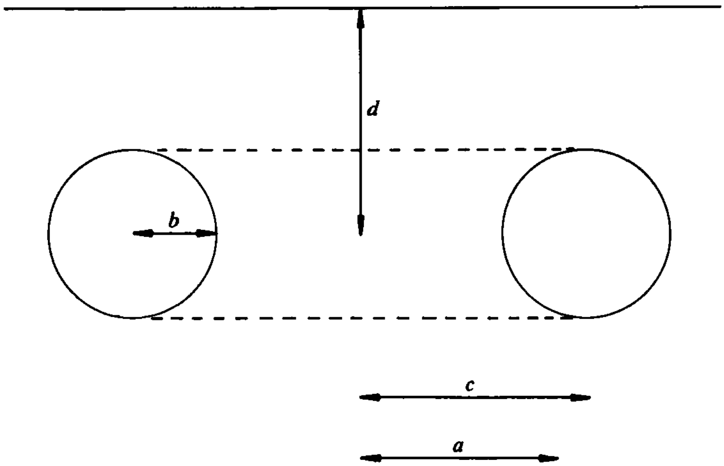

FIGURE 1. A cross-section through the submerged torus, showing the geometrical signficance of the parameters a, b, c, d.

where ∇^2 is the 3-dimensional Laplacian operator. We also assume that the fluid motion is small enough to allow us to use the linearized free-surface condition

$$K\phi + \frac{\partial \phi}{\partial y} = 0 \quad \text{on } y = 0, \tag{2.3}$$

where $K = \omega^2/g$ (g = acceleration due to gravity). The boundary condition at the surface of the torus is

$$\frac{\partial \phi}{\partial n} = U\frac{\partial y}{\partial n} \quad \text{on } \mathscr{S}. \tag{2.4}$$

The geometry of the problem is symmetrical about the y-axis, and so we expect that ϕ is independent of θ, i.e.

$$\phi = \phi(r, y) \text{ only.}$$

We also need to specify conditions to be satisfied at infinity. Far from the body we expect that the potential ϕ resembles that of a radially outgoing wave, and so we impose the *radiation condition* that

$$\phi(r, y) \sim \frac{A_0 g}{\omega}\frac{e^{K(ir-y)}}{(Kr)^{\frac{1}{2}}} + \text{smaller terms} \quad \text{as } r \rightarrow \infty, \tag{2.5}$$

where A_0 measures the amplitude of the waves at infinity. We also demand that the fluid motion vanishes as $y \rightarrow \infty$, i.e.

$$\nabla\phi \rightarrow 0 \quad \text{as } y \rightarrow \infty. \tag{2.6}$$

The equations (2.2)–(2.6) define a boundary-value problem for the known potential ϕ. To solve the problem, we proceed by defining toroidal coordinates (σ, ψ, θ) about the circle $\mathscr{C}$: $r = a$, $y = d$. Suppose P is a point in space and A, B are opposite ends of a diameter of $\mathscr{C}$ such that the plane APB contains the y-axis and B is closest to P. The coordinates (σ, ψ, θ) are then given by

$$\sigma = \ln\frac{AP}{BP}, \quad \psi = \pm A\hat{P}B \quad \text{for } y \gtrless d, \quad \theta \text{ as before.} \tag{2.7}$$

The surfaces σ = const, ψ = const, θ = const are mutually orthogonal. In particular, the 'level' surface σ = const describes the surface of a *torus*. This suggests that, by an appropriate choice of the value of a, $\mathscr{S}$ can be made to coincide with a level surface of the toroidal-coordinate system. In fact, if a is chosen so that

$$a = (c^2-b^2)^{\frac{1}{2}}, \tag{2.8}$$

then $\mathscr{S}$ is described as the torus $\sigma = \sigma_0$, where

$$\sinh\sigma_0 = \left(\frac{c^2}{b^2}-1\right)^{\frac{1}{2}}. \tag{2.9}$$

We also note that the boundary condition (2.4) on $\mathscr{S}$ can now be written as

$$\frac{\partial\phi}{\partial\sigma} = U\frac{\partial y}{\partial\sigma} \quad \text{on } \sigma = \sigma_0.$$

The relation between r, y and σ, ψ is

$$r = \frac{a\sinh\sigma}{\cosh\sigma-\cos\psi}, \quad y-d = \frac{a\sin\psi}{\cosh\sigma-\cos\psi}, \tag{2.10}$$

and the metric for this system is

$$\mathrm{d}s^2 = h^2\{\mathrm{d}\sigma^2+\mathrm{d}\psi^2+\sinh^2\sigma\,\mathrm{d}\theta^2\}, \quad h = \frac{a}{\cosh\sigma-\cos\psi}.$$

Fundamental quasiseparated solutions of Laplace's equation that are independent of θ are of the form

$$(\cosh\sigma-\cos\psi)^{\frac{1}{2}} \begin{Bmatrix}\cos n\psi\\ \sin n\psi\end{Bmatrix} \begin{Bmatrix}P_{n-\frac{1}{2}}(\cosh\sigma)\\ Q_{n-\frac{1}{2}}(\cosh\sigma)\end{Bmatrix} \tag{2.11}$$

(see Morse & Feshbach 1953, pp. 1301–1304). $P_{n-\frac{1}{2}}$, $Q_{n-\frac{1}{2}}$ are linearly independent solutions of Legendre's equation (of degree $n-\frac{1}{2}$) of the first and second kinds respectively, their definitions being taken as those used by Erdélyi *et al.* (1953, pp. 120–181). Functions of the first kind are associated with potentials that are singular on the circle $\mathscr{C}$ ($\sigma = \infty$) but bounded on the axis ($\sigma = 0$). Those of the second kind are associated with potentials bounded on $\mathscr{C}$ but singular on the axis. (In the subsequent work it is useful to regard $P_{n-\frac{1}{2}}(\cosh\sigma)$ as having a qualitative behaviour similar to that of $e^{+n\sigma}$, and $Q_{n-\frac{1}{2}}(\cosh\sigma)$ as behaving like $e^{-n\sigma}$, at least for $n, \sigma > 0$.) Finally we note that when n is an integer (2.11) describes potentials that are single-valued about $\mathscr{C}$.

3. Toroidal multipole potentials

We have seen that the surface $\mathscr{S}$ of the torus can be expressed as the level surface $\sigma = \sigma_0$ in a suitably chosen system of toroidal coordinates. Thus the boundary-value problem for the torus as defined by (2.2)–(2.6) is analogous to the 2-dimensional problem for a submerged circular cylinder as solved by Ursell (1950), in which the potential is expressed as an infinite sum of cylindrical *multipole potentials*. The corresponding problem for a submerged sphere has been treated by Srokosz (1979).

The purpose of this section is to construct multipole potentials in toroidal coordinates which are relevant to the problem of a submerged heaving torus. These toroidal multipole potentials individually satisfy the free-surface condition (2.3), the conditions at infinity (2.5), (2.6) and also have the property that they are singular on the circle $\mathscr{C}(\sigma = \infty)$ which lies inside the torus.

Fundamental solutions of Laplace's equation that are singular on $\mathscr{C}$ are of the form

$$(\cosh\sigma - \cos\psi)^{\frac{1}{2}} \cos n\psi \; P_{n-\frac{1}{2}}(\cosh\sigma), \tag{3.1a}$$

$$(\cosh\sigma - \cos\psi)^{\frac{1}{2}} \sin n\psi \; P_{n-\frac{1}{2}}(\cosh\sigma). \tag{3.1b}$$

The key step in constructing toroidal multipole potentials is to derive integral representations of (3.1), of the form

$$(\cosh\sigma - \cos\psi)^{\frac{1}{2}} \cos n\psi \; P_{n-\frac{1}{2}}(\cosh\sigma) = a \int_0^\infty C_n(\mu a)\, \mathrm{e}^{-\mu|y-d|} J_0(\mu r)\, \mathrm{d}\mu, \tag{3.2a}$$

$$(\cosh\sigma - \cos\psi)^{\frac{1}{2}} \sin n\psi \; P_{n-\frac{1}{2}}(\cosh\sigma) = \operatorname{sgn}(y-d)\, a \int_0^\infty S_n(\mu a)\, \mathrm{e}^{-\mu|y-d|} J_0(\mu r)\, \mathrm{d}\mu \quad \text{for } n = 0, 1, 2, 3, \ldots. \tag{3.2b}$$

The expressions on either side of (3.2) represent potentials that are symmetric about the axis $r = 0\,(\sigma = 0)$. Thus to determine the unknown functions $C_n(\mu a)$ and $S_n(\mu a)$ it is sufficient to compare values on the axis of symmetry, i.e.

$$(1-\cos\psi)^{\frac{1}{2}} \cos n\psi = a \int_0^\infty C_n(\mu a)\, \mathrm{e}^{-\mu|y-d|}\, \mathrm{d}\mu, \tag{3.3a}$$

$$(1-\cos\psi)^{\frac{1}{2}} \sin n\psi = \operatorname{sgn}(y-d)\, a \int_0^\infty S_n(\mu a)\, \mathrm{e}^{-\mu|y-d|}\, \mathrm{d}\mu. \tag{3.3b}$$

Let us consider just the first of these, (3.3a). On the axis we have, using (2.9),

$$y-d = \frac{a\sin\psi}{1-\cos\psi} = a\cot\tfrac{1}{2}\psi;$$

therefore

$$|\sin\tfrac{1}{2}\psi| = \frac{a}{[(y-d)^2+a^2]^{\frac{1}{2}}}. \tag{3.4}$$

Now

$$(1-\cos\psi)^{\frac{1}{2}} \cos n\psi = 2^{\frac{1}{2}} |\sin\tfrac{1}{2}\psi| \cos n\psi = 2^{\frac{1}{2}} |\sin\tfrac{1}{2}\psi| \sum_{m=0}^{n} \epsilon_m^n \sin^{2m}\tfrac{1}{2}\psi, \tag{3.5}$$

where

$$\epsilon_0^n = 1, \quad \epsilon_m^n = (-1)^m \frac{4n^2[4n^2-2^2]\ldots[4n^2-(2m-2)^2]}{(2m)!}$$

(see Gradshteyn & Ryzhik 1980, p. 28). We can use this last result to rewrite (3.3a) as

$$2^{\frac{1}{2}} \sum_{m=0}^{n} \epsilon_m^n \frac{a^{2m+1}}{[(y-d)^2+a^2]^{m+\frac{1}{2}}} = a \int_0^\infty C_n(\mu a)\, \mathrm{e}^{-\mu|y-d|}\, \mathrm{d}\mu.$$

Using the known result

$$\int_0^\infty \mu^m J_m(\mu a)\, \mathrm{e}^{-\mu p}\, \mathrm{d}\mu = \frac{(2a)^m (m-\frac{1}{2})!}{\pi^{\frac{1}{2}} (p^2+a^2)^{m+\frac{1}{2}}}$$

(see Watson 1944, p. 386), we deduce that the function $C_n(\mu a)$ is given by the *finite* sum

$$C_n(\mu a) = \sum_{m=0}^{n} \epsilon_m^n \frac{\pi^{\frac{1}{2}}}{(m-\frac{1}{2})!\, 2^{m-\frac{1}{2}}} (\mu a)^m J_m(\mu a), \quad n = 0, 1, 2, 3\ldots, \tag{3.6}$$

where the coefficients ϵ_m^n are defined by (3.5). To determine the function $S_n(\mu a)$ we treat (3.3b) in much the same way. The details of the analysis are very similar, and so we need only to state the final result:

$$S_n(\mu a) = n2^{\frac{1}{2}}\left\{2(\mu a)\,J_0(\mu a) + \sum_{m=1}^{n-1} \frac{\pi^{\frac{1}{2}}(-1)^m\,[4n^2-2^2]\ldots[4n^2-(2m)^2]}{2^m\,(m+\frac{1}{2})!\,(2m+1)!}(\mu a)^{m+1}\,J_m(\mu a)\right\}$$

$$\text{for } n = 1, 2, 3, \ldots. \quad (3.7)$$

We have now fully determined the functions $C_n(\mu a)$, $S_n(\mu a)$ which appear in the integral representations (3.2).

By analogy with the work of Thorne (1953), we now construct toroidal multipole potentials $\phi_n^{(1)}$ and $\phi_n^{(2)}$ by writing

$$\phi_n^{(1)} = (\cosh\sigma - \cos\psi)^{\frac{1}{2}} \cos n\psi\, P_{n-\frac{1}{2}}(\cosh\sigma) + a\int_0^\infty f_n^{(1)}(\mu)\,\mathrm{e}^{-\mu y}\,J_0(\mu r)\,\mathrm{d}\mu, \quad (3.8a)$$

$$\phi_n^{(2)} = (\cosh\sigma - \cos\psi)^{\frac{1}{2}} \sin n\psi\, P_{n-\frac{1}{2}}(\cosh\sigma) + a\int_0^\infty f_n^{(2)}(\mu)\,\mathrm{e}^{-\mu y}\,J_0(\mu r)\,\mathrm{d}\mu. \quad (3.8b)$$

The functions $f_n^{(1)}(\mu)$, $f_n^{(2)}(\mu)$ are chosen so that $\phi_n^{(1)}$, $\phi_n^{(2)}$ individually satisfy the free-surface condition (2.3). For example, using the integral representation (3.2a) it is easily verified that

$$\left[K + \frac{\partial}{\partial y}\right]_{y=0} \phi_n^{(1)} = a\int_0^\infty J_0(\mu a)\,[C_n(\mu a)\;\mathrm{e}^{-\mu d}(\mu+K) - f_n^{(1)}(\mu)\,(\mu-K)]\,\mathrm{d}\mu,$$

and so to satisfy the free-surface condition we require that

$$f_n^{(1)}(\mu) = \frac{\mu+K}{\mu-K}\,\mathrm{e}^{-\mu d}\,C_n(\mu a), \quad n = 0, 1, 2, \ldots.$$

By similar arguments we can show that

$$f_n^{(2)}(\mu) = \frac{-(\mu+K)}{\mu-K}\,\mathrm{e}^{-\mu d}\,S_n(\mu a), \quad n = 1, 2, 3, \ldots.$$

We must also ensure that $\phi_n^{(1)}$, $\phi_n^{(2)}$ individually satisfy the radiation condition (2.5) for each value of n. This is achieved by the familiar device of indenting the contour of integration in (3.8) so as to run *under* the simple pole of the integrands, at $\mu = K$. In fact, for this indented contour, it can be shown that

$$\phi_n^{(1)} \sim \pi\mathrm{i}(Ka)\,C_n(Ka)\,\mathrm{e}^{-K(y+d)}\,H_0^{(1)}(Kr) + \text{smaller terms},$$

$$\phi_n^{(1)} \sim -\pi\mathrm{i}(Ka)\,S_n(Ka)\,\mathrm{e}^{-K(y+d)}\,H_0^{(1)}(Kr) + \text{smaller terms} \quad \text{as } r \to \infty.$$

Let us now summarize the achievements of this section. We have constructed toroidal multipole potentials $\phi_n^{(1)}$, $\phi_n^{(2)}$ in the forms

$$\phi_n^{(1)} = (\cosh\sigma - \cos\psi)^{\frac{1}{2}} \cos n\psi\, P_{n-\frac{1}{2}}(\cosh\sigma) + a\,⨍_0^\infty \frac{\mu+K}{\mu-K}C_n(\mu a)\,\mathrm{e}^{-\mu(y+d)}\,J_0(\mu r)\,\mathrm{d}\mu, \quad (3.9a)$$

$$\phi_n^{(2)} = (\cosh\sigma - \cos\psi)^{\frac{1}{2}} \sin n\psi\, P_{n-\frac{1}{2}}(\cosh\sigma) - a\,⨍_0^\infty \frac{\mu+K}{\mu-K}S_n(\mu a)\,\mathrm{e}^{-\mu(y+d)}\,J_0(\mu r)\,\mathrm{d}\mu, \quad (3.9b)$$

where $⨍$ denotes that the contour of integration runs *under* the simple pole of the integrands, at $\mu = K$. By construction, these multipole potentials individually satisfy the conditions at the free surface and at infinity, and also have the property that they are singular on the circle $\mathscr{C}: r = a, y = d$ ($\sigma = \infty$), which lies *inside* the torus.

4. Mathematical solution of the potential problem

By an argument similar to that used by Gregory (1967), it can be shown that the velocity potential ϕ may be written as an infinite sum of toroidal multipole potentials, viz

$$\phi = Ua\left\{\sum_{n=0}^{\infty} \alpha_n \phi_n^{(1)} + \sum_{n=1}^{\infty} \beta_n \phi_n^{(2)}\right\}. \tag{4.1}$$

By construction, this expression for ϕ satisfies Laplace's equation in the fluid (2.2), the free-surface condition (2.3) and the conditions at infinity (2.5), (2.6). We now show that the coefficients $\{\alpha_n, \beta_n\}$ can be chosen so that ϕ satisfies the remaining boundary condition, on the torus itself, which is most conveniently stated in the form

$$(\cosh\sigma_0 - \cos\psi)^{\frac{1}{2}}\frac{\partial\phi}{\partial\sigma} = U(\cosh\sigma_0 - \cos\psi)^{\frac{1}{2}}\frac{\partial y}{\partial\sigma} \quad \text{on } \sigma = \sigma_0, \quad -\pi < \psi \leqslant \pi. \tag{4.2}$$

Now

$$y-d = \frac{a\sin\psi}{\cosh\sigma - \cos\psi} = a\frac{4\sqrt{2}}{\pi}(\cosh\sigma - \cos\psi)^{\frac{1}{2}}\sum_{m=1}^{\infty} m\sin m\psi\, Q_{m-\frac{1}{2}}(\cosh\sigma) \quad \text{for } \sigma > 0$$

(see Erdélyi *et al.* 1953, p. 166, (3)). It follows that

$$\begin{aligned}
&U(\cosh\sigma_0 - \cos\psi)^{\frac{1}{2}}\frac{\partial y}{\partial\sigma}\bigg|_{\sigma=\sigma_0} \\
&\quad= Ua\frac{4\sqrt{2}}{\pi}(\cosh\sigma_0 - \cos\psi)^{\frac{1}{2}}\left\{\frac{\frac{1}{2}\sinh\sigma_0}{(\cosh\sigma_0 - \cos\psi)^{\frac{1}{2}}}\sum_{m=1}^{\infty} m\sin m\psi\, Q_{m-\frac{1}{2}}(\cosh\sigma_0)\right. \\
&\qquad\left. + (\cosh\sigma_0 - \cos\psi)^{\frac{1}{2}}\sum_{m=1}^{\infty} m\sin m\psi \frac{\mathrm{d}}{\mathrm{d}\sigma}Q_{m-\frac{1}{2}}(\cosh\sigma)\bigg|_{\sigma=\sigma_0}\right\} \\
&\quad= Ua\frac{4\sqrt{2}}{\pi}\sum_{m=1}^{\infty}\gamma_m \sin m\psi,
\end{aligned} \tag{4.3}$$

where the coefficients $\{\gamma_m\}$ depend on σ_0 and are given by

$$\gamma_m = -\tfrac{1}{2}(m-1)\hat{q}_{m-1} + mq_m^* - \tfrac{1}{2}(m+1)\hat{q}_{m+1}, \tag{4.4}$$

and here we have employed the notation

$$\hat{q}_m = \frac{\mathrm{d}}{\mathrm{d}\sigma}Q_{m-\frac{1}{2}}(\cosh\sigma)\bigg|_{\sigma=\sigma_0}, \quad q_m^* = \tfrac{1}{2}\sinh\sigma_0\, Q_{m-\frac{1}{2}}(\cosh\sigma_0) + \cosh\sigma_0\,\hat{q}_m. \tag{4.5}$$

Thus the right-hand side of (4.2) can be expressed as a Fourier (sine) series in the angle ψ, over $-\pi < \psi \leqslant \pi$. Our next task is to derive a similar Fourier expansion of the left-hand side of (4.2), since we can then equate corresponding coefficients of $\cos m\psi$ and $\sin m\psi$, and hence obtain a system of equations involving the unknowns $\{\alpha_n, \beta_n\}$.

To proceed, we recall that the multipole potentials $\phi_n^{(1)}$ and $\phi_n^{(2)}$ are given by (3.9a) and (3.9b) respectively. The integrals in these expressions define 'image' potentials which are regular in the half-space $y > -d$. Thus they can be expanded as infinite series of toroidal harmonics which are *regular* around the circle $\mathscr{C}: r = a, y = d$ (i.e. $\sigma = \infty$), which lies inside the torus, i.e. we can write

$$\phi_n^{(1)} = (\cosh\sigma - \cos\psi)^{\frac{1}{2}}\Big\{\cos n\psi\, P_{n-\frac{1}{2}}(\cosh\sigma) + a_0^n\, Q_{-\frac{1}{2}}(\cosh\sigma) + \sum_{m=1}^{\infty}(a_m^n \cos m\psi + b_m^n \sin m\psi)\, Q_{m-\frac{1}{2}}(\cosh\sigma)\Big\}, \quad (4.6a)$$

$$\phi_n^{(2)} = (\cosh\sigma - \cos\psi)^{\frac{1}{2}}\Big\{\sin n\psi\, P_{n-\frac{1}{2}}(\cosh\sigma) + c_0^n\, Q_{-\frac{1}{2}}(\cosh\sigma) + \sum_{m=1}^{\infty}(c_m^n \cos m\psi + d_m^n \sin m\psi)\, Q_{m-\frac{1}{2}}(\cosh\sigma)\Big\}. \quad (4.6b)$$

Here the infinite series converge in the range $\sigma^* < \sigma < \infty$, where

$$\sigma^* = \tfrac{1}{2}\ln\left[1 + \frac{a^2}{d^2}\right], \quad (4.6c)$$

i.e. the series converge everywhere inside a toroidal annulus, 'centred' on the circle $\mathscr{C}: r = a, y = d$, which extends up to the 'image' circle $\mathscr{C}^*: r = a, y = -d$. In principle we can determine the coefficients $\{a_m^n, b_m^n, c_m^n, d_m^n\}$ by considering the Fourier expansions of the integral in (4.5) over *any* fixed torus $\sigma = \text{const}$ ($\sigma^* < \sigma < \infty$). By these means it can be shown that

$$a_m^n = [(1+\delta_{0,m})\, \pi\, Q_{m-\frac{1}{2}}(\cosh\sigma_0)]^{-1} \int_{-\pi}^{\pi} \frac{F_n^{(1)}(\sigma_0, \psi; Ka)}{(\cosh\sigma_0 - \cos\psi)^{\frac{1}{2}}} \cos m\psi \, \mathrm{d}\psi, \quad (4.7a)$$

$$b_m^n = [\pi\, Q_{m-\frac{1}{2}}(\cosh\sigma_0)]^{-1} \int_{-\pi}^{\pi} \frac{F_n^{(1)}(\sigma_0, \psi; Ka)}{(\cosh\sigma_0 - \cos\psi)^{\frac{1}{2}}} \sin m\psi \, \mathrm{d}\psi, \quad m = 0, 1, 2, \ldots, \quad (4.7b)$$

where $\delta_{i,j}$ is the Kronecker delta function ($= 1$ if $i = j$ and zero if $i \neq j$) and $F_n^{(1)}$ denotes the integral

$$F_n^{(1)} = a \,⨍_0^{\infty} \frac{\mu + K}{\mu - K} C_n(\mu a)\, \mathrm{e}^{-\mu(y+d)} J_0(\mu r)\, \mathrm{d}\mu. \quad (4.8a)$$

Equivalent expressions can be written down for the coefficients $\{c_m^n, d_m^n\}$ in terms of the function $F_n^{(2)}(\sigma_0, \psi; Ka)$, which is given by the integral

$$F_n^{(2)} = -a \,⨍_0^{\infty} \frac{\mu + K}{\mu - K} S_n(\mu a)\, \mathrm{e}^{-\mu(y+d)} J_0(\mu r)\, \mathrm{d}\mu. \quad (4.8b)$$

It is also clear that the coefficients $\{a_m^n, b_m^n, c_m^n, d_m^n\}$ can only depend on the values of σ^*, Ka and Kd.

Although it is possible to compute numerical approximations to the $\{a_m^n, b_m^n, c_m^n, d_m^n\}$ by means of evaluating integrals like (4.7), this is a rather expensive procedure to implement, owing to the fact that the expressions for the $F_n^{(i)}, i = 1, 2$, themselves need to be evaluated using quadrature techniques over a range of values of Ka. Fortunately, we can derive simple analytical expressions for the 'initial' coefficients $\{a_m^0, b_m^0, c_m^0, d_m^0\}$, and by using certain recurrence formulae it is possible to generate all of the 'higher' coefficients $\{a_m^n, b_m^n, c_m^n, d_m^n\}_{n\geqslant 1}$. (The details of this alternative method are outlined in the Appendix.) The use of these recurrence formulae is very efficient computationally, and gives a much higher accuracy than could be achieved by exploiting integral expressions like (4.7). Whichever method of evaluation is used, the important point is that the coefficients appearing in the expansions (4.5) can be considered as *known*.

We now use the expansions of $\phi_n^{(1)}, \phi_n^{(2)}$, given by (4.6), to evaluate the left-hand side of the boundary condition (4.2). After some manipulation, we can write the expression $(\cosh\sigma_0 - \cos\psi)^{\frac{1}{2}}(\partial\phi/\partial\sigma)|_{\sigma=\sigma_0}$ as a Fourier series in the angle ψ, and if we then compare the coefficients of $\cos m\psi$ and $\sin m\psi$ with those appearing in (4.3) we see that to satisfy the boundary condition on the body the (unknown) coefficients $\{\alpha_n, \beta_n\}$ must satisfy an infinite system of linear equations, with constant coefficients. The details of the analysis are rather laborious, and so we will merely state the final result, that

$$M_{mn} X_n = C_m, \quad m = 1,2,3,\ldots, \tag{4.9}$$

where

$$X_1 = \alpha_0, \quad X_2 = 0, \quad \left.\begin{aligned} X_{2n+1} &= \alpha_n, \\ X_{2n+2} &= \beta_n, \end{aligned}\right\} \quad n = 1,2,3,\ldots, \tag{4.10}$$

$$C_1 = C_2 = C_{2n+1} = 0, \quad C_{2n+2} = \gamma_n \text{ (known)}, \quad n = 1,2,3,\ldots, \tag{4.11}$$

and the matrix elements M_{mn} are given by

$$\left.\begin{aligned} M_{1,2n+1} &= q_0^* a_0^n - \tfrac{1}{2}\hat{q}_1 a_1^n + (p_0^* \delta_{0,n} - \tfrac{1}{2}\hat{p}_1 \delta_{1,n}), \\ M_{1,2n+2} &= q_0^* c_0^n - \tfrac{1}{2}\hat{q}_1 c_1^n, \end{aligned}\right\} \tag{4.12a}$$

$$\left.\begin{aligned} M_{2,2n+1} &= 0, \\ M_{2,2n+2} &= \delta_{0,n}, \end{aligned}\right\} \tag{4.12b}$$

$$\left.\begin{aligned} M_{3,2n+1} &= -\hat{q}_0 a_0^n + q_1^* a_1^n - \tfrac{1}{2}\hat{q}_2 a_2^n + (-\hat{p}_0 \delta_{0,n} + p_1^* \delta_{1,n} - \tfrac{1}{2}\hat{p}_2 \delta_{2,n}), \\ M_{3,2n+2} &= -\hat{q}_0 c_0^n + q_1^* c_1^n - \tfrac{1}{2}\hat{q}_2 c_2^n, \end{aligned}\right\} \tag{4.12c}$$

$$\left.\begin{aligned} M_{4,2n+1} &= q_1^* b_1^n - \tfrac{1}{2}\hat{q}_2 b_2^n, \\ M_{4,2n+2} &= q_1^* d_1^n - \tfrac{1}{2}\hat{q}_2 d_2^n + (p_1^* \delta_{1,n} - \tfrac{1}{2}\hat{p}_2 \delta_{2,n}), \end{aligned}\right\} \tag{4.12d}$$

$$\left.\begin{aligned} M_{2m+1,2n+1} &= -\tfrac{1}{2}\hat{q}_{m-1} a_{m-1}^n + q_m^* a_m^n - \tfrac{1}{2}\hat{q}_{m+1} a_{m+1}^n \\ &\quad + (-\tfrac{1}{2}\hat{p}_{m-1} \delta_{m-1,n} + p_m^* \delta_{m,n} - \tfrac{1}{2}\hat{p}_{m+1} \delta_{m+1,n}), \\ M_{2m+1,2n+2} &= -\tfrac{1}{2}\hat{q}_{m-1} b_{m-1}^n + q_m^* b_m^n - \tfrac{1}{2}\hat{q}_{m+1} b_{m+1}^n, \end{aligned}\right\} \tag{4.12e}$$

$$\left.\begin{aligned} M_{2m+2,2n+1} &= -\tfrac{1}{2}\hat{q}_{m-1} b_{m-1}^n + q_m^* b_m^n - \tfrac{1}{2}\hat{q}_{m+1} b_{m+1}^n, \\ M_{2m+2,2n+2} &= -\tfrac{1}{2}\hat{q}_{m-1} d_{m-1}^n + q_m^* d_m^n - \tfrac{1}{2}\hat{q}_{m+1} d_{m+1}^n, \\ &\quad + (-\tfrac{1}{2}\hat{p}_{m-1} \delta_{m-1,n} + p_m^* \delta_{m,n} - \tfrac{1}{2}\hat{p}_{m+1} \delta_{m+1,n}) \end{aligned}\right\} \tag{4.12f}$$

for $m = 2,3,4,\ldots, \quad n = 0,1,2,\ldots.$

In these equations we have used the expressions $\hat{q}_m$ and q_m^* defined earlier, and also the analogous expressions $\hat{p}_m$ and p_m^* given by

$$\hat{p}_m = \frac{\mathrm{d}}{\mathrm{d}\sigma} P_{m-\frac{1}{2}}(\cosh\sigma)\bigg|_{\sigma=\sigma_0}, \quad p_m^* = \tfrac{1}{2}\sinh\sigma_0\, P_{m-\frac{1}{2}}(\cosh\sigma_0) + \cosh\sigma_0\, \hat{p}_m.$$

In principle, the infinite system of equations (4.9) can now be solved to give the values of the coefficients $\{\alpha_n, \beta_n\}$. In this way we have now constructed the *exact* solution to the problem, in the form of an expansion for ϕ given by (4.1). In practice, of course, we can only solve finite systems of equations and hence obtain an *approximation* to the true solution — this procedure is now discussed in §5.

5. Calculation of the added-mass and damping coefficients $A(Ka)$, $B(Ka)$

In an engineering context, the important quantity to calculate for this 'heave-radiation' problem is the vertical force F exerted on the torus by the fluid. If the fluid motion is time-harmonic then so is F, and it is convenient to write

$$F = \mathrm{Re}\{f\mathrm{e}^{-\mathrm{i}\omega t}\}, \tag{5.1}$$

where f is a complex-valued force coefficient. When presenting numerical results it is conventional to 'scale' the force with respect to the mass of the fluid displaced by the torus ($= \rho 2\pi^2 a^3 \cosh\sigma_0/\sinh^3\sigma_0$) and its maximum acceleration $U\omega$, and we write

$$f = -\mathrm{i}Ua^3\rho\omega 2\pi^2 \frac{\cosh\sigma_0}{\sinh^3\sigma_0}(A+\mathrm{i}B). \tag{5.2}$$

Here A is called the (dimensionless) 'added-mass coefficient' and measures the component of the force F that is in phase with the acceleration of the torus; B is known as the (dimensionless) 'damping coefficient' and measures the component of the force in phase with the velocity. For a given torus, at a fixed depth under the free surface, A and B are functions of Ka only.

In a linear theory, the pressure p in the fluid, in excess of its hydrostatic value, is given by

$$p = -\rho\frac{\partial\Phi}{\partial t} = \rho\,\mathrm{Re}\{\mathrm{i}\omega\phi\,\mathrm{e}^{-\mathrm{i}\omega t}\},$$

and by integrating this pressure over the surface of the torus we have that

$$f = \rho\omega\mathrm{i}\iint_{\text{torus}} \phi\frac{\partial y}{\partial n}\,\mathrm{d}S.$$

Using (5.2) and (2.10) we deduce that

$$\begin{aligned} A+\mathrm{i}B &= \left[-\mathrm{i}Ua^3\rho\omega\, 2\pi^2\frac{\cosh\sigma_0}{\sinh^3\sigma_0}\right]^{-1} \\ &\quad \times\rho\omega\mathrm{i}\int_0^{2\pi}\int_{-\pi}^{\pi}\phi(\sigma_0,\psi)\left\{\frac{\sin\psi\sinh\sigma_0}{\cosh\sigma_0-\cos\psi}\right\}\frac{a^2\sinh\sigma_0}{(\cosh\sigma_0-\cos\psi)^2}\,\mathrm{d}\psi\,\mathrm{d}\theta \\ &= -(Ua\pi)^{-1}\operatorname{sech}\sigma_0\sinh^5\sigma_0\int_{-\pi}^{\pi}\phi(\sigma_0,\psi)\frac{\sin\psi}{(\cosh\sigma_0-\cos\psi)^3}\,\mathrm{d}\psi. \end{aligned} \tag{5.3}$$

We now use the expansion for ϕ, given by (4.1), and the results

(i) $\displaystyle\int_{-\pi}^{\pi}\frac{\cos m\psi\sin\psi}{(\cosh\sigma_0-\cos\psi)^{\frac{5}{2}}}\,\mathrm{d}\psi = 0$ (integrand is an odd function of ψ),

(ii) $\displaystyle\int_{-\pi}^{\pi}\frac{\sin m\psi\sin\psi}{(\cosh\sigma_0-\cos\psi)^{\frac{5}{2}}}\,\mathrm{d}\psi = \frac{2m}{3}\int_{-\pi}^{\pi}\frac{\cos m\psi}{(\cosh\sigma_0-\cos\psi)^{\frac{3}{2}}}\,\mathrm{d}\psi$ (by parts)

$$= -\frac{8\sqrt{2}\,m}{3\sinh\sigma_0}\hat{q}_m$$

to replace the integral in (5.3) by an infinite series,† and it can be verified that

† In (ii) we have used the identity

$$Q_{m-\frac{1}{2}}(\cosh\sigma) = \frac{1}{2\sqrt{2}}\int_{-\pi}^{\pi}\frac{\cos m\psi}{(\cosh\sigma-\cos\psi)^{\frac{1}{2}}}\,\mathrm{d}\psi$$

(see Erdélyi *et al.* 1953, p. 166, (3)).

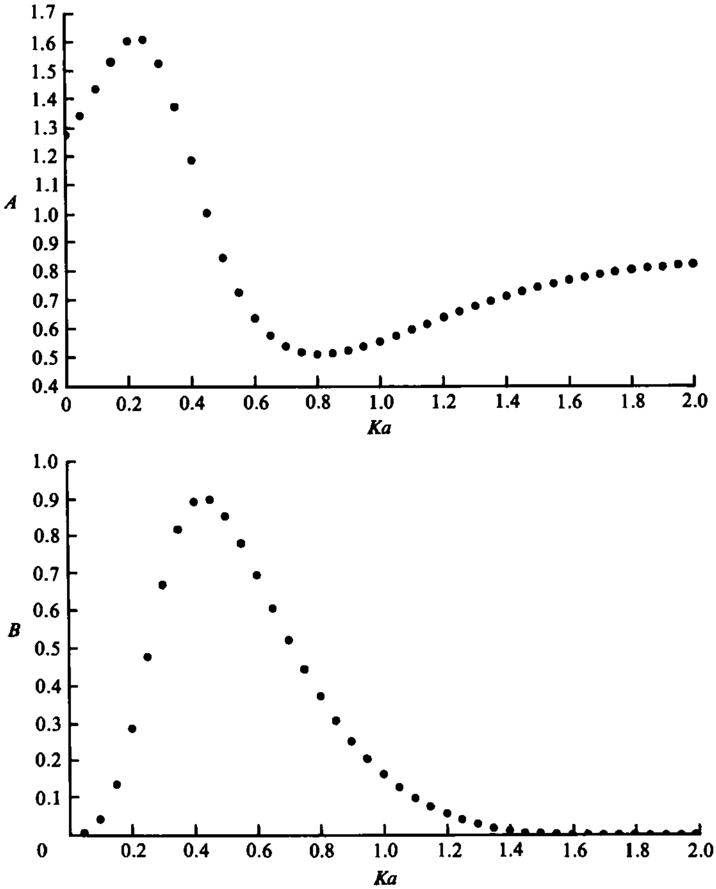

FIGURE 2. The added-mass and damping coefficients of a submerged torus for which $c/b = \frac{3}{2}$, $d/b = 2$ (giving $\sigma_0 = 0.96$).

$$A + \mathrm{i}B = \frac{8\sqrt{2}}{3\pi}\frac{\sinh^4\sigma_0}{\cosh\sigma_0}\sum_{m=1}^{\infty} m\hat{q}_m\left\{\beta_m p_m + q_m\left[\sum_{n=0}^{\infty}\alpha_n b_m^n + \sum_{n=1}^{\infty}\beta_n d_m^n\right]\right\}, \tag{5.4}$$

where we have employed the notation

$$p_m = P_{m-\frac{1}{2}}(\cosh\sigma_0), \quad q_m = Q_{m-\frac{1}{2}}(\cosh\sigma_0),$$

and the related functions $\hat{p}_m$, $\hat{q}_m$ have been defined previously.

In principle, the $\{\alpha_m, \beta_m\}$ are found by solving the infinite system of linear equations (4.9). In practice, however, we can only solve *finite* systems of equations, and for the purpose of computation it is natural to truncate (4.9) to a $(2N+2)\times(2N+2)$ system and hence attempt to calculate *approximations* to the finite set of coefficients $\{\alpha_m, \beta_m\}, 0 \leqslant m \leqslant N$. These are then used to give approximations to the added-mass and damping coefficients A and B by truncating the infinite series in (5.4) after the first N terms. (An obvious check on the validity of this procedure is to choose successively larger values of N and verify that the corresponding values for A and B converge to definite limits.) Computational experience shows that the value of N that should be chosen so as to achieve a certain accuracy depends very strongly on σ_0 (i.e. the geometry of the torus) and to a lesser extent on the value of Ka. More specifically, as σ_0 increases (i.e. the torus becomes 'slender'), N decreases, for a fixed value of Ka.

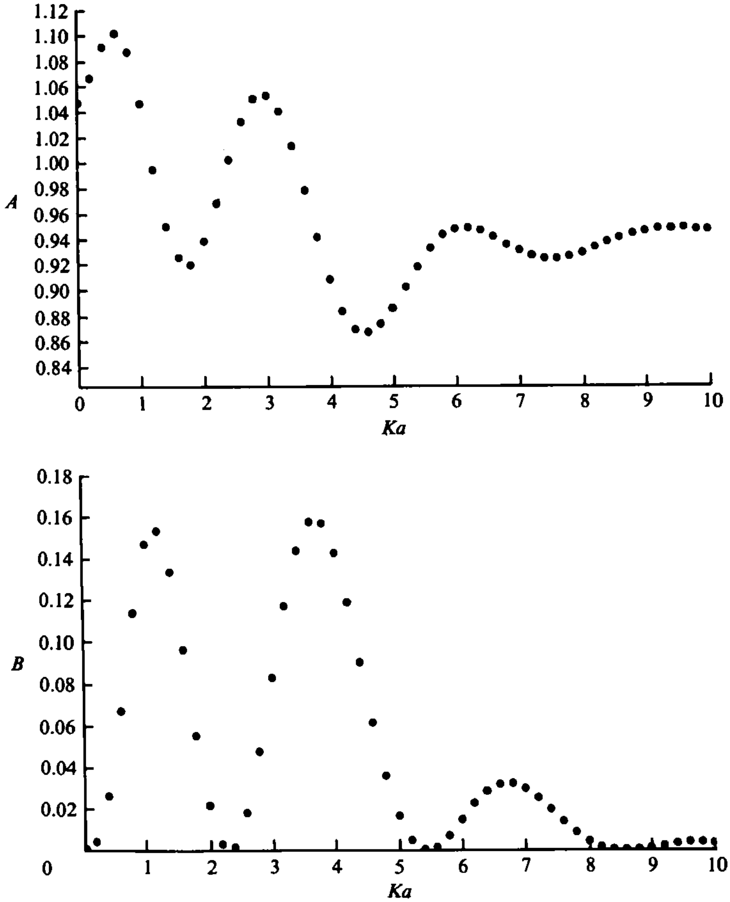

FIGURE 3. The added-mass and damping coefficients of a submerged 'slender' torus for which $b = 5.3$ m, $c = 44.7$ m, $d = 20$ m (giving $a = 44.4$ m, $\sigma_0 = 2.82$).

In this paper we present numerical results for two different toroidal geometries. Figure 2 shows the graphs of A and B for a torus that has $c/b = \frac{3}{2}, b/d = 2$ (giving $\sigma_0 = \cosh^{-1}\frac{3}{2} = 0.962\ldots$). For this configuration the choice $N = 15$ appears to produce values for A and B that have an absolute accuracy $\approx \pm 10^{-8}$ over the range $0 \leqslant Ka \leqslant 5$. (We recall that $a = (c^2 - b^2)^{\frac{1}{2}}$.) By comparison, figure 3 shows the results for a torus that has $b = 5.3$ m, $c = 44.7$ m, $d = 20$ m (giving $a = 44.4$ m, $\sigma_0 = 2.82$); these are the appropriate values for the proposed RS-35 design of semisubmersible platform, working at its operational depth. It is clear that this torus is 'slender' in the sense that the aspect ratio $b/2c$ is much less than unity. For this case, taking $N = 8$ was sufficient to produce results with an absolute accuracy $\approx \pm 10^{-8}$.

Comparing figures 2 and 3, it is clear that the results for a slender torus have a different character than those for a non-slender torus. In particular, the author's computations suggest that the damping coefficients of slender tori very nearly vanish in the neighbourhood of the points $Ka = 2.4$, 5.5, 8.7, 11.8 etc.† In §6 we discuss the

† Those familiar with the properties of Bessel functions will notice that these values of Ka lie suspiciously close to the first few zeroes of $J_0(Ka)$. In fact, it was this observation which prompted the detailed asymptotic analysis presented in §6.

asymptotic solution of (4.9), in the limit $b/2c \to 0$ (i.e. $e^{-\sigma_0} \to 0$), and show that it is possible to obtain good closed-form approximations for the added-mass and damping coefficients of slender tori.

6. The asymptotic form of the added-mass and damping coefficients of slender tori

If the torus is slender, in the sense that its 'tubular radius' is small compared with its 'overall diameter', then from (2.1) we have that

$$\frac{b}{2c} \ll 1. \tag{6.1}$$

We recall that in our chosen system of toroidal coordinates $\{\sigma, \psi, \theta\}$ the surface of the torus is given by $\sigma = \sigma_0$, where

$$\sinh \sigma_0 = \left[\frac{c^2}{b^2} - 1\right]^{\frac{1}{2}},$$

and it is easily verified that

$$e^{-\sigma_0} \sim \frac{b}{2c} \quad \text{as } \frac{b}{2c} \to 0. \tag{6.2}$$

Thus, to investigate the hydrodynamic characteristics of slender tori in waves, it is natural to consider the asymptotic solution of the linear equations (4.9) in the limit $e^{-\sigma_0} \to 0$. We will now demonstrate that this approach leads to good closed-form approximations to the added-mass and damping coefficients of a slender torus. When using these approximations it is important to note that σ_0 need not be very large in order that $e^{-\sigma_0} \ll 1$: For example, the RS-35 design of semisubmersible has $b/2c \approx \frac{1}{17}$, giving $\sigma_0 \approx 2.82$ and $e^{-\sigma_0} \approx 0.06$.

The principal step in the analysis is to modify the system of equations (4.9) in such a way that the matrix becomes strongly diagonally dominant as $e^{-\sigma_0} \to 0$. To do this we first note that as $e^{-\sigma_0} \to 0$

$$Q_{m-\frac{1}{2}}(\cosh \sigma_0) \sim \pi^{\frac{1}{2}} \frac{(m-\frac{1}{2})!}{m!} e^{-(m+\frac{1}{2})\sigma_0}, \tag{6.3a}$$

$$P_{m-\frac{1}{2}}(\cosh \sigma_0) \sim \begin{cases} \dfrac{2}{\pi} \sigma_0 e^{-\frac{1}{2}\sigma_0} & (m = 0), \\[2ex] \dfrac{(m-1)!}{\pi^{\frac{1}{2}}(m-\frac{1}{2})!} e^{(m-\frac{1}{2})\sigma_0} & (m > 0) \end{cases} \tag{6.3b}$$

(see Olver 1974, pp. 169–185). Using these results it is possible to calculate the asymptotic form of the quantities $\hat{q}_m$, q_m^*, $\hat{p}_m$, p_m^* and γ_m (and hence M_{mn}, C_m) which appear in §4. The details of this asymptotic analysis are easily obtained, but too numerous to present, and so we will confine our attention to the important points: The quantities $\hat{q}_m$, q_m^*, γ_m become exponentially small as $e^{-\sigma_0} \to 0$, whereas those in $\hat{p}_m$, p_m^* become exponentially large. More specifically,

$$Q_{m-\frac{1}{2}}(\cosh \sigma_0),\ \hat{q}_m = O[e^{-(m+\frac{1}{2})\sigma_0}], \quad \gamma_m,\ q_m^* = O[e^{-(m-\frac{1}{2})\sigma_0}],$$

$$P_{m-\frac{1}{2}}(\cosh \sigma_0),\ \hat{p}_m = O[e^{(m-\frac{1}{2})\sigma_0}], \quad p_m^* = O[e^{(m+\frac{1}{2})\sigma_0}]$$

as $e^{-\sigma_0} \to 0$, at least for $m > 0$. These results can now be used to deduce the asymptotic nature of the matrix coefficients M_{mn}, as defined in (4.12), and the 'right-hand sides' C_m, given by (4.11). In this way we find that, for the infinite system of equations (4.9), i.e.

$$M_{mn} X_n = C_m,$$

as $e^{-\sigma_0} \to 0$ the matrix $\boldsymbol{M}$ develops a definite banded structure in which the elements contained in the five diagonals nearest to the leading diagonal are asymptotically dominant over the other elements of the matrix. Unfortunately, the asymptotic structure of the matrix $\boldsymbol{M}$ is still sufficiently complicated to prevent us from reaching any useful conclusions about the properties of the solution vector $\boldsymbol{X}$ as $e^{-\sigma_0} \to 0$.

The next step in the asymptotic analysis is to exploit the fact that $\boldsymbol{X}$ is defined by (4.10) to be the vector

$$\boldsymbol{X} = (\alpha_0, 0, \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots)^{\mathrm{t}},$$

and we recall that the $\{\alpha_n, \beta_n\}$ are the unknown coefficients appearing in the infinite-series representation of the potential ϕ, given by (4.1). Now, by an argument analogous to that used by Gregory (1967), it can be established that these infinite series must converge when $\sigma = \sigma_0$, that is, on the surface of the torus. It follows that

$$\alpha_n P_{n-\frac{1}{2}}(\cosh \sigma_0), \quad \beta_n P_{n-\frac{1}{2}}(\cosh \sigma_0) \to 0 \quad \text{as } n \to \infty,$$

and we deduce that the $\{\alpha_n, \beta_n\}$ become exponentially small as $n \to \infty$. This suggests that we should 'rescale' the solution vector $\boldsymbol{X}$ by writing

$$\boldsymbol{X}^* = (\alpha_0 P_{-\frac{1}{2}}(\cosh \sigma_0),\ 0,\ \alpha_1 P_{\frac{1}{2}}(\cosh \sigma_0),\ \beta_1 P_{\frac{1}{2}}(\cosh \sigma_0),\ \alpha_2 P_{\frac{3}{2}}(\cosh \sigma_0),\ \beta_2 P_{\frac{3}{2}}(\cosh \sigma_0), \ldots)^{\mathrm{t}}, \tag{6.4}$$

where we still have that $X_n^* \to 0$ as $n \to \infty$. This rescaled vector $\boldsymbol{X}^*$ satisfies the infinite system of equations

$$M_{mn}^* X_n^* = C_m, \tag{6.5}$$

where the matrix elements M_{mn}^* are given by

$$M_{m,\,2n+1}^* = \frac{M_{m,\,2n+1}}{P_{n-\frac{1}{2}}(\cosh \sigma_0)}, \quad M_{m,\,2n+2}^* = \frac{M_{m,\,2n+2}}{P_{n-\frac{1}{2}}(\cosh \sigma_0)}. \tag{6.6}$$

If we now examine the asymptotic form of the matrix $\boldsymbol{M}^*$ as $e^{-\sigma_0} \to 0$ we find that the leading diagonal contains elements of order e^{σ_0} while all of the other elements are $O(1)$ or less. If we apply a further transformation, to normalize the elements on the leading diagonal, we obtain a final system of equations in the form

$$[\boldsymbol{I} + \boldsymbol{A}]\boldsymbol{X}^* = \boldsymbol{C}^*, \tag{6.7}$$

$$C_1^* = C_2^* = C_{2n+2}^* = 0, \quad C_{2n+2}^* = \frac{\gamma_n}{p_n^*} P_{n-\frac{1}{2}}(\cosh \sigma_0), \quad n = 1, 2, 3, \ldots. \tag{6.8}$$

Here $\boldsymbol{I}$ is the (infinite) identity matrix and $\boldsymbol{A}$ is an (infinite) matrix in which every element is $O(e^{-\sigma_0})$ or less. In the limit $e^{-\sigma_0} \to 0$ we expect that the asymptotic solution of (6.7) takes the form

$$\boldsymbol{X}^* \sim [\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{A}^2 - \boldsymbol{A}^3 \ldots]\, \boldsymbol{C}^*. \tag{6.9}$$

Using the relations (6.4), (6.8), (6.9) and the asymptotic results (6.3), we can now deduce the asymptotic form of the coefficients $\{\alpha_n, \beta_n\}$ as $e^{-\sigma_0} \to 0$. In fact, a careful analysis reveals that

$$\left.\begin{aligned} &\alpha_0, \alpha_1 = O(\mathrm{e}^{-4\sigma_0}), \quad \alpha_n = O(\mathrm{e}^{-(2n+1)\sigma_0}) \quad (n \geqslant 2), \\ &\mathrm{Re}\,(\beta_1) = -\sqrt{2}\,\pi\,\mathrm{e}^{-2\sigma_0} + O(\mathrm{e}^{-3\sigma_0}), \quad \beta_n = O(\mathrm{e}^{-2n\sigma_0}) \quad (n \geqslant 2), \\ &\mathrm{Im}\,(\beta_1) = -\frac{\pi^2}{\sqrt{2}}\,\mathrm{e}^{-\frac{3}{2}\sigma_0}\,\mathrm{Im}\,(d_1^1), \end{aligned}\right\} \tag{6.10}$$

where d_1^1 is a complex-valued coefficient first introduced in (4.6*b*). The asymptotic results given above can now be used, in conjunction with (5.4), to produce an asymptotic estimate for the damping coefficient $B(Ka)$. The dominant terms in the infinite series for $B(Ka)$ are those involving β_1, and using (5.4) it can be verified that

$$B(Ka) \sim \tfrac{1}{2}\pi^2\,\mathrm{Im}\,(d_1^1)\,\mathrm{e}^{-2\sigma_0} + O(\mathrm{e}^{-3\sigma_0}) \quad \text{as } \mathrm{e}^{-\sigma_0} \to 0. \tag{6.11}$$

Now

$$\begin{aligned} \mathrm{Im}\,(d_1^1) &= -4(Ka)\,\mathrm{Im}\,(c_0^1) \\ &= \frac{4\sqrt{2}}{\pi}(Ka^2)\,\mathrm{Im}\,⨍_0^\infty \frac{\nu+K}{\nu-K}\,\mathrm{e}^{-2\nu d}\,S_0(\nu a)\,J_0(\nu a)\,\mathrm{d}\nu \\ &= 8\sqrt{2}\,(Ka)^2\,\mathrm{e}^{-2kd}\,S_0(Ka)\,J_0(Ka) \\ &= 32(Ka)^3\,\mathrm{e}^{-2kd}\,\{J_0(Ka)\}^2, \end{aligned}$$

where we have used the relations given in the Appendix and the expression for $S_0(Ka)$ given by (3.7). Hence it follows from (6.11) that the damping coefficient for a slender torus has the asymptotic form

$$B(Ka) \sim [16\pi^2\,\mathrm{e}^{-2\sigma_0}]\,(K\alpha)^3\,\mathrm{e}^{-2Ka(d/a)}\{J_0(Ka)\}^2 + O(\mathrm{e}^{-3\sigma_0}) \quad \text{as } \mathrm{e}^{-\sigma_0} \to 0. \tag{6.12}$$

It is also possible to use (5.4), (6.10) to produce an asymptotic estimate for the added mass $A(Ka)$ of a slender torus, and to leading order we find that

$$A(Ka) = 1 + O(\mathrm{e}^{-2\sigma_0}) \quad \text{as } \mathrm{e}^{-\sigma_0} \to 0.$$

In principle it is possible to obtain the next term in this asymptotic expansion of $A(Ka)$, but this would involve a very long and intricate calculation. A more pragmatic approach is to employ the so-called 'Kramers–Kronig relations', as discussed by Kotik & Mangulis (1962). In the present notation the Kramers–Kronig relations take the form

$$A(Ka) - A(0) = \frac{Ka}{\pi}⨍_0^\infty \frac{B(z)\,\mathrm{d}z}{z(z-Ka)}, \tag{6.13a}$$

$$B(Ka) = \frac{(Ka)^{\frac{1}{2}}}{\pi}⨍_0^\infty \frac{A(z)-A(\infty)}{z^{\frac{1}{2}}(Ka-z)}\,\mathrm{d}z, \tag{6.13b}$$

where $⨍$ denotes the Cauchy principal-value integral. If we replace the term $B(z)$ in the relation (6.13*a*) by the first term in its asymptotic expansion as $\mathrm{e}^{-\sigma_0} \to 0$, given by (6.12), we find that

$$A(Ka) \sim A(0) + [16\pi\,\mathrm{e}^{-2\sigma_0}]\,Ka⨍_0^\infty z^2\,\mathrm{e}^{-2z(d/a)}\,J_0^2(z)\frac{\mathrm{d}z}{z-Ka} + O(\mathrm{e}^{-3\sigma_0}). \tag{6.14}$$

This last result can be interpreted in the following way. If we know the value of the added mass of a slender torus at zero frequency (i.e. $Ka = 0$) then (6.14) gives an approximation to the added mass $A(Ka)$ at all other frequencies (i.e. $0 < Ka < \infty$).

In practice the asymptotic results (6.12), (6.14) have been found to give excellent approximations to the values of the added-mass and damping coefficients of slender

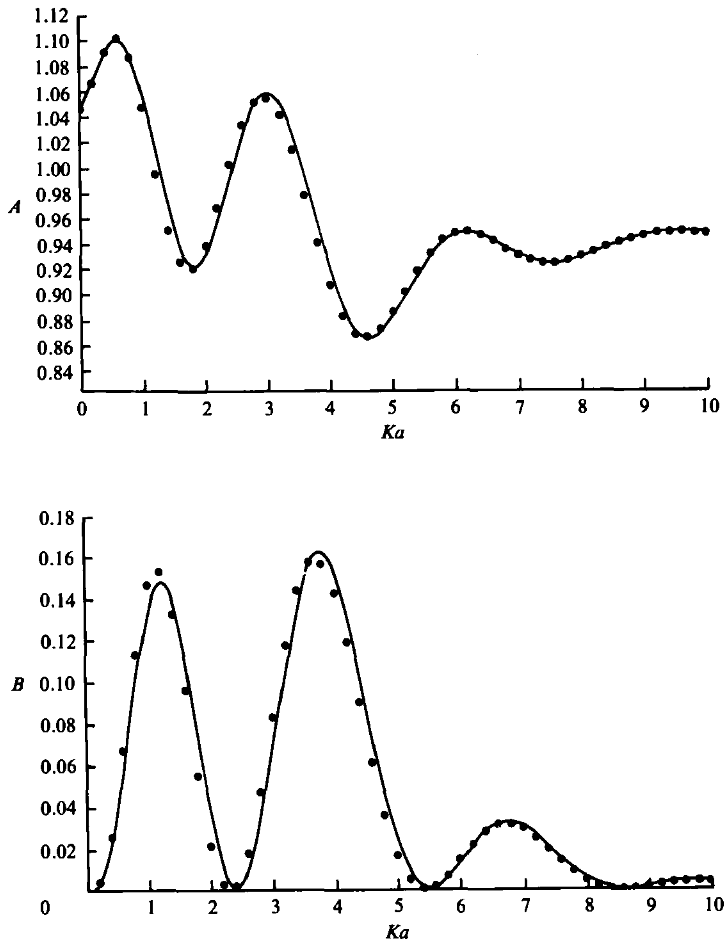

FIGURE 4. A comparison of the computed results for a 'slender' torus with those predicted by the asymptotic theory as $e^{-\sigma_0} \to 0$: ●, previously computed values (as shown in figure 3); —, asymptotic approximations given by (6.12), (6.14).

tori that are sufficiently well submerged.† For example, figure 4 compares the closed-form approximations to $A(Ka)$, $B(Ka)$ with the 'exact' values for the RS-35 slender torus presented at the end of §5. The asymptotic form of $B(Ka)$, given by (6.12), suggests that the damping coefficient of a slender torus takes values very close to zero whenever the value of $J_0(Ka)$ is close to zero, i.e. $Ka \approx 2.4, 5.5, 8.7$ etc., and this is clearly illustrated in figure 4.

† *It is possible to quantify this statement about the necessary depth of submergence.* A more detailed investigation of the matrix elements M_{mn} in (4.9) reveals that the asymptotic arguments presented in §6 are more accurately based on the assumption that $e^{-(\sigma_0-\sigma^*)} \ll 1$, where σ^* is a parameter depending on the depth, and is given by (4.6*b*). Now

$$e^{-(\sigma_0-\sigma^*)} \ll 1 \Rightarrow \frac{b}{2c}\left[1+\frac{a^2}{d^2}\right]^{\frac{1}{2}} \ll 1,$$

and so for our asymptotic results to apply we must have $b/2c \ll 1$ *and that* $[1+a^2/d^2]^{\frac{1}{2}}$ *is not large.* Physically, this means that the depth of submergence should not be very small when compared with the overall diameter of the torus.

Finally, it should be noted that the asymptotic results obtained for a slender torus can be derived by an alternative method in which the dominant part of the velocity potential is related to a ring-dipole whose moment is derived by analogy with the problem of a circular cylinder in two dimensions. In this way the integral in (6.14) and the corresponding 'residue' term of (6.12) are associated with the free-surface contribution to the expression for a submerged vertically aligned dipole, after the necessary circular integration.

7. Discussion

A simple application of Green's theorem to the boundary-value problem stated in §2 shows that the amplitude of the waves radiated to infinity by the heaving torus is related to $[B(Ka)]^{\frac{1}{2}}$, the square root of its damping coefficient (see Newman 1962). Thus for a slender torus the results of §6 suggest that the wave amplitude at infinity tends to vanish around each discrete frequency for which $J_0(Ka) = 0$.

The general methods described in this paper could also be used to treat the physically distinct, but mathematically similar, problem of the diffraction of a train of plane waves by a fixed submerged torus. Since the torus has a vertical axis of symmetry the formulation of this diffraction problem is the same as that for a general body of revolution, as discussed by Hulme (1983) and others.

From an engineering viewpoint, an important quantity to calculate for the diffraction problem is F, the magnitude of the net vertical component of the (time-harmonic) wave-induced force exerted on a fixed, submerged torus. Now F can be derived from $B(Ka)$, the damping coefficient in the corresponding heave-radiation problem, by using the 'Haskind relations' discussed by Newman (1962). We find that

$$F = 2\pi(Aa^2\rho g)\left[\frac{\cosh\sigma_0}{\sinh^3\sigma_0}\right]^{\frac{1}{2}}\left[\frac{B(Ka)}{Ka}\right]^{\frac{1}{2}},$$

where A is the amplitude of the incident plane waves. For a torus that is slender, in the sense that $b/2c \ll 1$, we can replace $B(Ka)$ by the first term in its asymptotic expansion as $e^{-\sigma_0} \to 0$, given by (6.12), and we deduce that

$$|F| = (Aa^2\rho g)\,(16\pi^2 e^{-2\sigma_0})\,Ka\, e^{-Ka(d/a)}\,|J_0(Ka)| + O(e^{-3\sigma_0}) \quad \text{as } e^{-\sigma_0} \to 0.$$

Thus we see that the net vertical force on a fixed slender torus in waves tends to vanish around the discrete set of frequencies for which $J_0(Ka) = 0$. This property may be of some importance in regard to the future design of ring-hulled semisubmersible platforms.

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Appendix

Let us consider the expansion of the 'image' potential

$$a\int_0^\infty \frac{\nu+K}{\nu-K} C_n(\nu a)\, e^{-\nu(y+d)}\, J_0(\nu r)\, d\nu \quad (= \chi_n, \text{ say}).$$

This potential is regular about the circle $\mathscr{C}$: $r = a, y = d$, and so has an expansion, in toroidal coordinates $\{\sigma, \psi, \theta\}$ defined about $\mathscr{C}$, of the form

$$\chi_n = (\cosh\sigma - \cos\psi)^{\frac{1}{2}}\Big\{a_0^n\, Q_{-\frac{1}{2}}(\cosh\sigma) + \sum_{m=1}^{\infty} (a_m^n \cos m\psi + b_m^n \sin m\psi)\, Q_{m-\frac{1}{2}}(\cosh\sigma)\Big\}. \quad \text{(A 1)}$$

Our aim now is to determine the coefficients $\{a_m^n, b_m^n\}$ appearing in this expansion. Some of the early coefficients are easily obtained. Following a method described by Hulme (1981), in which we take the formal limits $\sigma \to \infty$ on both sides of (A 1), we find that

$$a_0^n = \frac{\sqrt{2}}{\pi} a \int_0^\infty \frac{\nu+K}{\nu-K}\, e^{-2\nu d}\, C_n(\nu a)\, J_0(\nu a)\, d\nu. \quad \text{(A 2)}$$

Similarly, if we set $y = d$ in (A 1), and then take the d/dr derivative of both sides, in the limit $\sigma \to \infty$ we find that

$$a_1^n = 2a_0^n - \frac{4\sqrt{2}}{\pi} a^2 \int_0^\infty \frac{\nu(\nu+K)}{\nu-K}\, e^{-2\nu d}\, C_n(\nu a)\, J_1(\nu a)\, d\nu \quad \text{(A 3)}$$

(again see Hulme 1981). The integrals in (A 2), (A 3) can successfully be evaluated using numerical quadrature, providing that d/a is not too small.

The fundamental step in determining the $\{a_m^n, b_m^n\}$ is to exploit the fact that

$$\left(K + \frac{\partial}{\partial y}\right)\chi_n = -a \int_0^\infty (\nu+K)\, e^{-\nu(y+d)}\, C_n(\nu a)\, J_0(\nu r)\, d\nu$$
$$= -\left(K - \frac{\partial}{\partial y}\right) a \int_0^\infty e^{-\nu(y+d)}\, C_n(\nu a)\, J_0(\nu r)\, d\nu. \quad \text{(A 4)}$$

Let us, for the moment, assume that we know the expansion coefficients $\{\xi_m^n, \eta_m^n\}$ associated with the integral in (A 4), viz.

$$a\int_0^\infty e^{-\nu(y+d)}\, C_n(\nu a)\, J_0(\nu r)\, d\nu = (\cosh\sigma - \cos\psi)^{\frac{1}{2}}\Big\{\xi_0^n\, Q_{-\frac{1}{2}}(\cosh\sigma) + \sum_{m=1}^{\infty} (\xi_m^n \cos m\psi + \eta_n^m \sin m\psi)\, Q_{m-\frac{1}{2}}(\cosh\sigma)\Big\}. \quad \text{(A 5)}$$

If we apply the operator $K + \partial/\partial y$ to both sides of (A 1) and use (A 4), (A 5) it can be established that

$$2(Ka)\, a_m^n + (m-\tfrac{1}{2})\, b_{m-1}^n - 2mb_m^n + (m+\tfrac{1}{2})\, b_{m+1}^n = \frac{\sqrt{2}}{\pi}\Big\{-2(Ka)\,\xi_m^n + (m-\tfrac{1}{2})\,\eta_{m-1}^n - 2m\eta_m^n + (m+\tfrac{1}{2})\,\eta_{m+1}^n\Big\},$$

$$2(Ka)\, b_m^n - (m-\tfrac{1}{2})\, a_{m-1}^n + 2ma_m^n - (m+\tfrac{1}{2})\, a_{m+1}^n = \frac{\sqrt{2}}{\pi}\Big\{-2(Ka)\,\eta_m^n - (m-\tfrac{1}{2})\,\xi_{m-1}^n + 2m\xi_m^n - (m+\tfrac{1}{2})\,\xi_{m+1}^n\Big\},$$

for $m \geqslant 2$, together with

$$\left.\begin{aligned} 2(Ka)\, a_1^n - 2b_1^n + \tfrac{3}{2}b_2^n &= \frac{+\sqrt{2}}{\pi}\Big\{-2(Ka)\,\xi_1^n - 2\eta_1^n + \tfrac{3}{2}\eta_2^n\Big\},\\ 2(Ka)\, b_1^n - a_0^n + 2a_1^n - \tfrac{3}{2}a_2^n &= \frac{\sqrt{2}}{\pi}\Big\{-2(Ka)\,\eta_1^n - \xi_0^n + 2\xi_1^n - \tfrac{3}{2}\eta_2^n\Big\},\\ 2(Ka)\, a_0^n + \tfrac{1}{2}b_1^n &= \frac{+\sqrt{2}}{\pi}\Big\{-2(Ka)\,\xi_0^n + \tfrac{1}{2}\eta_1^n\Big\} \end{aligned}\right\} \quad \text{(A 6)}$$

(the proof of this result uses the same general techniques as described previously by Hulme (1981).

The relations (A 6) together with the 'initial values' given by (A 3) define a set of *recurrence formulae* which completely determine the coefficients $\{a_m^n, b_m^n\}$. The expansion of the integral

$$a\int_0^\infty \frac{\nu+K}{\nu-K} S_n(\nu a)\, \mathrm{e}^{-\nu(y+d)} J_0(\nu r)\, \mathrm{d}\nu$$

can be treated in an analogous manner.

Our next task is actually to determine the coefficients $\{\xi_m^n, \eta_m^n\}$ appearing in the expressions (A 5), (A 6). From (3.6) we see that $C_0(\nu a) = \sqrt{2}\, J_0(\nu a)$, and so for $n = 0$ the integral in (A 5) represents the potential due to a uniform distribution of sources around the 'image' circle $\mathscr{C}^*: r = a, y = -d$. The expansion of this ring-source potential about the circle $\mathscr{C}: r = a, y = d$ has been derived by Hulme (1981), and we have that

$$a\int_0^\infty \mathrm{e}^{-\nu(y+d)} C_0(\nu a) J_0(\nu r)\, \mathrm{d}\nu = \frac{\sqrt{2}}{\pi}(\cosh\sigma - \cos\psi)^{\frac{1}{2}} (\cosh\sigma^* - \cos\psi^*)^{\frac{1}{2}} \times \left\{ \sum_{m=0}^{\infty} \epsilon_m \cos m(\psi-\psi^*)\, P_{m-\frac{1}{2}}(\cosh\sigma^*)\, Q_{m-\frac{1}{2}}(\cosh\sigma) \right\} \quad (\sigma^* < \sigma < \infty), \quad \text{(A 7)}$$

where $\{\sigma^*, \psi^*\}$ relate to the position of the image circle $\mathscr{C}^*$ and are given by

$$\sigma^* = \tfrac{1}{2}\ln\left\{1+\frac{d^2}{a^2}\right\}, \quad \psi^* = \tan^{-1}\frac{a}{d}.$$

Thus

$$\left.\begin{aligned} \xi_m^0 &= \epsilon_m(\cosh\sigma^* - \cos\psi^*)^{\frac{1}{2}} \cos m\psi^*\, P_{n-\frac{1}{2}}(\cosh\sigma^*), \\ \eta_m^0 &= \epsilon_m(\cosh\sigma^* - \cos\psi^*)^{\frac{1}{2}} \sin m^*\, P_{n-\frac{1}{2}}(\cosh\sigma^*), \end{aligned}\right\} \quad \text{(A 8)}$$

where ϵ_m is Neuman's factor. Now, from (3.6),

$$C_1(\nu a) = \sqrt{2}\, J_0(\nu a) - 2\sqrt{2}\,(\nu a) J_1(\nu a),$$

and we notice that

$$a\int_0^\infty \mathrm{e}^{-\nu(y+d)} C_1(\nu a) J_0(\nu a)\, \mathrm{d}\nu = \left(2a\frac{\partial}{\partial a} - 1\right) a\int_0^\infty \mathrm{e}^{-\nu(y+d)} C_0(\nu a) J_0(\nu r)\, \mathrm{d}\nu.$$

Hence to determine the coefficients $\{\xi_m^1, \eta_m^1\}$, we apply the operator $2a\,\partial/\partial a - 1$ to both sides of (A 7), and after some manipulation we find that

$$\left.\begin{aligned} \xi_m^1 &= -(m+\tfrac{1}{2})\,\xi_{m+1}^0 + (m-\tfrac{1}{2})\,\xi_{m-1}^0 + 2a\frac{\partial \xi_m^0}{\partial a}, \\ \eta_m^1 &= -(m+\tfrac{1}{2})\,\eta_{m+1}^0 + (m-\tfrac{1}{2})\,\eta_{m-1}^0 + 2a\frac{\partial \eta_m^0}{\partial a} \end{aligned}\right\} \quad (m \geqslant 2),$$

and

$$\xi_1^1 = \xi_0^0 - \tfrac{3}{2}\xi_2^0 + 2a\frac{\partial \xi_1^0}{\partial a}, \quad \eta_1^1 = -\tfrac{3}{2}\eta_2^0 + 2a\frac{\partial \eta_1^0}{\partial a}, \quad \xi_0 = -\tfrac{1}{2}\xi_1^0 + 2a\frac{\partial \xi_0^0}{\partial a},$$

where here the derivatives $2a\,\partial\xi_m^0/\partial a$, $2a\,\partial\xi_m^0/\partial a$ can be found in closed form as functions of σ^* and ψ^*. In a very similar manner, it is possible to derive 3-term recursion formulae that generate the remaining coefficients $\{\xi_m^n, \eta_m^n\}, n \geqslant 2, m \geqslant 0$, by successive application of the operator $2a\,\partial/\partial y$ to both sides of (A 5), followed by a suitable rearrangement of the terms. The details of this calculation are left as an exercise for the reader.

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